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RESEARCH ARTICLE

No arbitrage conditions for simple trading strategies

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Abstract Strict local martingales may admit arbitrage opportunities with respect to the class of simple trading strategies. (Since there is no possibility of using doubling strategies in this framework, the losses are not assumed to be bounded from below.) We show that for a class of non-negative strict local martingales, the strong Markov property implies the no arbitrage property with respect to the class of simple trading strategies. This result can be seen as a generalization of a similar result on three dimensional Bessel process in Delbaen and Schachermayer (Math Finance 4:343–348, 1994). We also provide no arbitrage conditions for stochastic processes within the class of simple trading strategies with shortsale restriction.

Keywords Simple trading strategies · Arbitrage · Sticky processes · Shortsales restriction

JEL Classification G10

1 Introduction

We consider a market with a risky asset with price process X and a risk-free asset with price process B in the time horizon $[0, \infty)$. We will assume that $B_t \equiv 1$, which

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corresponds to taking the bond price as the numéraire. The price processes are defined on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ satisfying the "usual hypotheses" (i.e., the filtration $\mathbb{F}$ is right continuous, and $\mathcal{F}_0$ contains all of the P null sets of $\mathcal{F}$). In this paper, we restrict our trading strategies to the following class of simple integrands:

Definition 1 The set of simple trading strategies with short sale restriction is given by $S^0(\mathbb{F}) = \{g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} : n \geq 2, 0 \leq \tau_1 \leq \cdots \leq \tau_n$ where all of the τ_j are bounded $\mathbb{F}$-stopping times; g_0 is a non-negative real number,and the g_j are non-negative real valued $\mathcal{F}_{\tau_i}$ measurable random variables$\}$. The set of simple trading strategies is denoted by $S(\mathbb{F})$ and is given by $S(\mathbb{F}) = S^0(\mathbb{F}) - S^0(\mathbb{F})$.

Let $K^0 = \{(H \cdot X)_\infty | H \in S^0(\mathbb{F})\}$, $K^s = \{(H \cdot X)_\infty | H \in S(\mathbb{F})\}$ denote the outcomes of the corresponding trading strategies respectively for any adapted price process X.

Definition 2 We say X satisfies the no arbitrage property with respect to $S^0(\mathbb{F})$, $S(\mathbb{F})$ separately if $K^0 \cap L_0^+ = \{0\}$, $K^s \cap L_0^+ = \{0\}$, respectively. (Here L_0^+ is the collection of equivalence classes of non-negative measurable functions on the probability space $(\Omega, \mathcal{F}, P)$.)

Observe that we do not assume that the losses are bounded from below, which is not necessary since the class of doubling strategies is not a subset of either $S(\mathbb{F})$ or $S^0(\mathbb{F})$. But as it was observed in Delbaen and Schachermayer (1994b) strict local martingales may admit arbitrage opportunities with respect to $\mathbb{S}(F)$. In Sect. 2, we give a necessary and sufficient conditions, which we baptize as "condition $(*)$", for local martingales to admit no arbitrage with respect to $S(\mathbb{F})$. We give examples of strict local martingales that do/do not admit arbitrage with respect to $S(\mathbb{F})$. We also provide an alternative proof of Theorem 6 of Delbaen and Schachermayer (1995) [see also Proposition 2.1 in Delbaen and Schachermayer (1994b)] as an application of our no arbitrage characterization of local martingales. We also show that condition $(*)$ is preserved under composition with non-decreasing functions and give an application of this result.

In Sect. 3, we derive a sufficient and necessary condition for no-arbitrage, and show that the no arbitrage property is preserved under composition with strictly increasing functions. As an application of this result we can construct processes that are not semi-martingales but do not admit arbitrage with respect to $S^0(\mathbb{F})$.

2 No arbitrage conditions wrt $S(\mathbb{F})$

In this section we provide a necessary and sufficient condition for the no arbitrage property of non-negative *strict local martingales* [i.e., not true martingales, see, e.g., Delbaen and Schachermayer (1995)] with respect to the simple trading strategies $S(\mathbb{F})$. A typical example of a strict local martingale is the inverse process of three dimensional Bessel process, see Delbaen and Schachermayer (1994b). Strict local martingales appear in a number of ways in applications and some of their financial applications were discussed in Delbaen and Schachermayer (1994b, 1995, 2006).

Recently, strict local martingales were employed in modeling financial bubbles, see Jarrow et al. (2009b) and Protter and Pal (2008). Also see Delbaen and Shirakawa (2002), Heath and Schweizer (2000) and Sin (1998) for stock price models with stochastic volatility for which the most natural candidates for the pricing measures is an equivalent strict local martingale measure.

It is well known that local martingales satisfy the no arbitrage property when the admissible trading strategies are such that accumulated losses are bounded below Delbaen and Schachermayer (1994a). This condition on the losses is added to avoid doubling strategies. However, the collection of doubling strategies is not a subset the class of simple trading strategies $S(\mathbb{F})$. Therefore, we do not impose lower bound for the gain process in our definition of no arbitrage with simple trading strategies. As a result strict local martingales may admit arbitrage possibilities. Here we will give necessary and sufficient no arbitrage conditions for strict local martingales.

One of the advantages of working with $S(\mathbb{F})$ is that, the no arbitrage property of a process X with respect to $S(\mathbb{F})$ implies the no arbitrage property of the process $f(X)$ with respect to $S(\mathbb{F})$ as well, for any strictly increasing or strictly decreasing function f, see Corollary 5 in Jarrow et al. (2009a). However the same is not true for more general admissible strategies. For example, three dimensional Bessel process admits arbitrage with respect to the class of general admissible trading strategies, see Delbaen and Schachermayer (1995), while its inverse process, a non-negative strict local martingale, does not. This has the undesirable effect of having numéraire dependency in the existence of arbitrage [see Section 4 of Delbaen and Schachermayer (1995)].

To state the main result of this section, we introduce a condition that is weaker than stickiness [see, e.g., Bayraktar and Sayit (2008) and Guasoni (2006)]. We will show that this condition is a necessary and sufficient condition for the no arbitrage property with respect to $S(\mathbb{F})$ of non-negative strict local martingales. This characterization will allow us to state a general result on non-negative local martingales with strong Markov property.

Definition 3 We say that an adapted Càdlàg process X satisfies condition $(\star)$ with respect to the filtration $\mathbb{F}$ if for any bounded stopping time τ and any $A \in \mathcal{F}_\tau$ with $P(A) > 0$ we have

$$P\left(A \cap \left\{\inf_{t\in[\tau,T]} (X_t - X_\tau) > -\epsilon\right\}\right) > 0$$

for any $\epsilon > 0$ and T with $\tau \leq T$ a.s.

Remark 1 The condition $(\star)$ is a weaker condition than the sticky condition, compare Definition 2.9 of Guasoni (2006) with Definition 3. In Guasoni (2006), it was shown that strong Markov processes with regular points and continuous processes with full support on the space of continuous functions are sticky. As a result all these processes satisfy condition $(\star)$.

We are now ready to state the main result of this section:

Proposition 1 *Assume X is a nonnegative Càdlàg $\mathbb{F}$-semimartingale and X admits an equivalent local martingale measure Q. Then, X satisfies the no arbitrage property in $S(\mathbb{F})$ if and only if X satisfies the condition $(\star)$.*

Proof Sufficiency: Assume X satisfies the condition $(\star)$ and that there exists arbitrage opportunities. Since X is bounded below and admits an equivalent local martingale measure, it is a supermartingale under this equivalent measure. Without loss of generality we can assume its arbitrage strategy is of the form $-1_{(\tau_0,\tau_1]}$ for two bounded stopping times $\tau_0 \leq \tau_1$ [see Lemma 5 of Jarrow et al. (2009a)]. So we assume $X_{\tau_0} \geq X_{\tau_1}$ a.s. and $P(X_{\tau_0} > X_{\tau_1}) > 0$. Let K be a number such that the event $A = \{X_{\tau_0} < K\} \cap \{\tau_1 > \tau_0\}$ satisfies $P(A \cap \{X_{\tau_0} > X_{\tau_1}\}) > 0$. Note that there is such a K because the event $\{X_{\tau_0} > X_{\tau_1}\}$ has positive probability and $\{X_{\tau_0} > X_{\tau_1}\} \subset \{\tau_1 > \tau_0\}$ and that X is a càdlàgsuper-martingale, which implies that X is a.s. bounded on a compact domain see, e.g., Theorem 1.3.8 of Karatzas and Shreve (1991). Also we have that $A \in \mathcal{F}_{\tau_0}$. Fix any number T with $T \geq \tau_1$ a.s. and define the following two stopping times

$$\tau_0^A = \begin{cases} \tau_0 & \text{if } \omega \in A, \\ T & \text{if } \omega \notin A \end{cases}$$

and

$$\tau_1^A = \begin{cases} \tau_1 & \text{if } \omega \in A, \\ T & \text{if } \omega \notin A. \end{cases}$$

Let $\tau = \inf\{t \geq \tau_0^A : X_t > K + 1\} \wedge \tau_1^A$. In what follows we use N to denote sets of measure zero. (Although these sets might differ we use the same letter to avoid notational crowding.) If $\tau = \tau_1^A$ on $A - N$ then since $X_{\tau_0} \leq K$ and $X_{\tau_1} \leq K$ on $A - N$ we have that the local martingale $1_A(X_t - X_{\tau_0})$ is bounded in $[\tau_0, \tau_1]$. So it is a martingale in $[\tau_0, \tau_1]$. This contradicts with $P(A \cap \{X_{\tau_0} > X_{\tau_1}\}) > 0$. So we assume the event $B = A \cap \{\tau < \tau_1^A\}$ has positive probability. Note that $B \in \mathcal{F}_\tau$. Since X is Càdlàg, $X_\tau \geq K + 1$ on $B - N$. Since $X_{\tau_1} \leq K$ on A–N we have that $P(B \cap \{\inf_{t\in[\tau,T]}(X_t - X_\tau) > -\frac{1}{2}\}) = 0$. This contradicts with condition $(\star)$. So X satisfies the no arbitrage property in $S(\mathbb{F})$.

Necessity: Assume X has no arbitrage strategy in $S(\mathbb{F})$ and X does not satisfy condition $(\star)$. Then there is a bounded stopping time τ and $A \in \mathcal{F}_\tau$ with $P(A) > 0$ and $T, \epsilon > 0$ with $T \geq \tau$ a.s. such that $P(A \cap \{\inf_{t\in[\tau,T]}(X_t - X_\tau) > -\epsilon\}) = 0$. So $\inf_{t\in[\tau,T]}(X_t - X_\tau) \leq -\epsilon$ on A/N for a measure zero set N. Let $\tau^A = \tau$ on A and $\tau^A = T$ on the complement of A. Then τ^A is a stopping time. If we define $\theta = \inf\{t \geq \tau^A : (X_t - X_{\tau^A}) < -\frac{\epsilon}{2}\}$, then since X has right continuous paths $(X_\theta - X_{\tau^A}) \leq -\frac{\epsilon}{2}$ on A/N and also we have $\theta \leq T$ on A/N. Let $\theta^A = \theta$ on A and $\theta^A = T$ on the complement of A, then $-1_{(\tau^A,\theta^A]}$ is an arbitrage strategy for X in $S(\mathbb{F})$. This completes the proof. □

Remark 2 We should mention that Proposition 1 is useful only for strict local martingales. This is because all martingales satisfy condition $(\star)$.

Remark 3 We remark that the second part of the proof of the above proposition shows that if X satisfies the no arbitrage property with respect to $S(\mathbb{F})$, then X satisfies the condition $(\star)$. This fact will be used in the proof of Corollary 3 below.

Remark 4 We remark that the condition in Proposition 1 that X is non-negative can be relaxed. We can replace this condition by the requirement that the negative part of X belongs to class DL, a class of processes X such that for each $a > 0$, $\{X_S\}$ is uniformly integrable over all bounded stopping times $S \leq a$. We can do this replacement thanks to Proposition 2.2 of Elworthy et al. (1999), which shows that the local martingales whose negative part are of class DL are supermartingales.

An immediate corollary of Proposition 1 is the following general result on nonnegative semimartingales with strong Markov property.

Corollary 1 *Let X be a non-negative càdlàg $\mathbb{F}$-semimartingale that admits an equivalent local martingale measure. If X is sticky, then it satisfies the no arbitrage property in $S(\mathbb{F})$. Especially if X has strong Markov property* (*under the original measure*) *and if all points of X are regular, then X satisfies the no arbitrage property with respect to $S(\mathbb{F})$.*

Proof If S is sticky, then it satisfies the condition $(\star)$ and Proposition 1 applies [compare Definition 3 with Definition 2 of Bayraktar and Sayit (2008) and also see Proposition 1 of Bayraktar and Sayit (2008)]. If X has strong Markov property and all of its points are regular, then it is sticky [see Guasoni (2006)] and again Proposition 1 applies when X is strictly positive. □

Remark 5 Note that a strictly positive process X satisfies the no-arbitrage property with respect to $S(\mathbb{F})$ if and only if $1/X$ satisfies the no-arbitrage property with respect to $S(\mathbb{F})$. (This was observed in Delbaen and Schachermayer (1995) but can also be proved using Corollary 5 in Jarrow et al. (2009a).) Therefore X can be replaced by $1/X$ in Proposition 1 and Corollary 3.

Note that not all non-negative local martingales satisfy the condition $(\star)$, the following is an example, given in Delbaen and Schachermayer (1994b), of a non-negative process that admits an equivalent local martingale measure, but has arbitrage in $S(\mathbb{F})$.

Example 1 Let $(B_t)_{t\geq 0}$ be a Brownian motion with $B_0 = 1$. Let $\tau = \inf\{t > 0 : B_t = 0\}$. It is well know that τ is a.s. finite. Let

$$S_t = \begin{cases} B_{tan(\frac{t\pi}{2})\wedge\tau} & \text{when } t < 1, \\ B_\tau = 0 & \text{when } t = 1. \end{cases}$$

Then S is a nonnegative process that admits an equivalent local martingale measure, see, e.g., Delbaen and Schachermayer (1994b). The arbitrage strategy, H, of the process S is given by $H = -1$ on $(0, 1]$. Clearly $(H \cdot S)_1 = 1$ a.s. and $(H \cdot S)_0 = 0$ a.s. Observe that S does not satisfy the condition $(\star)$: we can let $A = \Omega$, $\tau = 0$, $T = 1$, $\epsilon = 0.5$ and obtain $P(\inf_{t\in[0,1]}\{t : (S_t - 1) > 0.5\}) = 0$.

A typical example of a non-negative local martingale that is not true martingale and that has strong Markov property is the inverse process of three dimensional Bessel process [see, e.g., Delbaen and Schachermayer (1995)]. The following is another example of a strict local martingale that satisfies the no arbitrage property in $S(\mathbb{F})$.

Example 2 Consider the CEV model

$$dX_t = aX_t dt + bX_t^{\rho} dB_t.$$

It is well known that the CEV models admit an equivalent martingale measure when $\rho \in (0, 1]$ and they admit a strict local martingale measure when $\rho > 1$, see, e.g., Delbaen and Shirakawa (2002). CEV models are regular strong Markov processes: for the case $\rho > 1$ this is stated in Heath and Schweizer (2000), on the other hand for $\rho \in (0, 1)$ this fact follows from Sections 5.4 and 5.5 of Karatzas and Shreve (1991) (since the conditions ND' and LI' on p. 343 hold). Finally observe that $\rho = 1$ is the classical geometric Brownian motion model. As a result CEV processes are sticky and they satisfy condition $(\star)$ (see Corollary 3) and therefore do not admit arbitrage possibilities with respect to $S(\mathbb{F})$ by Proposition 1.

In fact Delbaen and Shirakawa (2002) considered a more general class of processes of the form

$$dX_t = \sigma(X_t) dW_t,$$

and gave necessary and sufficient conditions on the function σ for these processes to be martingales/strict local martingales [see Theorem 1.6 of Delbaen and Shirakawa (2002)]. By assuming that σ also satisfies the ND' and LI' conditions mentioned above we obtain a large class of strict local martingales that admit no arbitrage with respect to $S(\mathbb{F})$.

The following corollary is the content of the Proposition 2.1 in Delbaen and Schachermayer (1994b) and Theorem 6 of Delbaen and Schachermayer (1995). Here we restate it and show a simple proof as an application of Proposition 1.

Corollary 2 *Let X be the Bessel process of dimension $\delta \in (2, \infty)$ or Bes(δ) (see Revuz and Yor (1999) Definition 11.1.1), and $X_0 > 0$. Then X and $1/X$ satisfy the no arbitrage property with respect to $S(\mathbb{F})$.*

Proof The process $M_t = X^{2-\delta}$ is a nonnegative local martingale [see Revuz and Yor (1999) Chapter 11, Exercise 1.16] and a regular strong Markov process. By Proposition 2 of Bayraktar and Sayit (2008) M has the sticky property and so it satisfies the condition $(\star)$. Now applying Proposition 1, we see that M satisfies the no arbitrage property with respect to $S(\mathbb{F})$. Since X is a composition of M with strictly decreasing function $f(x) = x^{-\frac{1}{\delta-2}}$ on $(0, \infty)$, from Corollary 5 in Jarrow et al. (2009a), which states that the no-arbitrage property for simple integrands is invariant under composition with strictly decreasing functions, the result for X follows. The result for $1/X$ follows from Remark 5. □

Remark 6 When X is a Bes(3), $1/X$ is a strict local martingale and therefore does not allow for arbitrage strategies with respect to general admissible strategies. However, X itself does admit arbitrage opportunities with respect to general admissible strategies (Delbaen and Schachermayer 1995). In order to prove this result (Delbaen and Schachermayer 1995) shows that all strictly positive martingales can be obtained as a reciprocal of a martingale as an h-transform. Protter and Pal (2008) uses this result to analyze the bubbles in European option prices when the stock price is given by $1/X$.

The following proposition shows that the property $(\star)$ is invariant under composition with a continuous nondecreasing function.

Proposition 2 *Let X be a stochastic process adapted to the filtration $\mathbb{F}$ and takes values in the interval (a, b) (including the cases when $a = -\infty$ and/or $b = \infty$). If X satisfies condition $(\star)$, then for any continuous nondecreasing function f defined on (a, b), the process $f(X)$ also satisfies condition $(\star)$.*

Proof First we assume a, b are bounded. We need to show for any bounded stopping time τ and any $A \in \mathcal{F}_\tau$ with $P(A) > 0$, we have $P(A \cap \{\inf_{t\in[\tau,T]}(f(X_t) - f(X_\tau)) > -\epsilon\}) > 0$ for any $\epsilon > 0$ and T with $\tau \le T$ a.s. Since X_τ takes values on (a, b) and $P(A) > 0$, for sufficiently large $n_0 \in \mathbb{N}$, the event $B = A \cap \{X_\tau \in [a + \frac{1}{n_0}, b - \frac{1}{n_0}]\} \in \mathcal{F}_\tau$ has positive probability. The function f is continuous on (a, b), therefore it is uniformly continuous on $[a + \frac{1}{n_0}, b - \frac{1}{n_0}]$. That is for a given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$ and $x, y \in [a + \frac{1}{n_0}, b - \frac{1}{n_0}]$ we have $|f(x) - f(y)| < \epsilon$. Since X satisfies condition $(*)$ we have that $P(B \cap \{\inf_{t\in[\tau,T]}(X_t - X_\tau) > -\delta\}) > 0$. Since f is uniformly continuous and non-decreasing we have $B\cap\{\inf_{t\in[\tau,T]}(X_t-X_\tau) > -\delta\} \subset B\cap\{\inf_{t\in[\tau,T]}(f(X_t)-f(X_\tau)) > -\epsilon\}$. So $P(A \cap \{\inf_{t\in[\tau,T]}(f(X_t) - f(X_\tau)) > -\epsilon\}) > 0$ since $B \subset A$. When $a = -\infty$ and/or $b = \infty$, the above proof can be adjusted by replacing $a + 1/n_0$ with $a_n \downarrow -\infty$ and/or $b - 1/n_0$ with $b_n \uparrow \infty$. □

An application of Proposition 2 is the following result:

Corollary 3 *Let X be a continuous semimartingale and assume X admits an equivalent local martingale measure Q. Then $X_t - \frac{1}{2}[X, X]_t$ admits no arbitrage with respect to $S(\mathbb{F})$ if and only if $X_t - \frac{1}{2}[X, X]_t$ satisfies condition $(\star)$.*

Proof If $X_t - \frac{1}{2}[X, X]_t$ has no arbitrage in $S(\mathbb{F})$ then $X_t - \frac{1}{2}[X, X]_t$ satisfies the condition $(\star)$ (see Remark 3 above). Conversely, if $X_t - \frac{1}{2}[X, X]_t$ satisfies the condition $(\star)$ then, by Proposition 2 above, $e^{X_t-\frac{1}{2}[X,X]_t}$ also satisfies the condition $(\star)$. Since $e^{X_t-\frac{1}{2}[X,X]_t}$ is a local martingale under the measure Q, by Proposition 1 it satisfies no arbitrage with respect to $S(\mathbb{F})$. Therefore by Corollary 5 of Jarrow et al. (2009a), $X_t - \frac{1}{2}[X, X]_t$ also satisfies no arbitrage with respect to $S(\mathbb{F})$. □

3 No arbitrage with shortsales restrictions

Let us denote $L^0_{++}(\Omega, \mathcal{F}, \mathbb{P}) := \{\eta \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : P(\eta \ge 0) = 1 \text{ and } P(\eta > 0) > 0\}$.

We begin with the characterization of no arbitrage with respect to $S^0(\mathbb{F})$.

Proposition 3 *An adapted càdlàg process $X_t, t \in [0, \infty)$ satisfies the no arbitrage property in $S^0(\mathbb{F})$ if and only if for any two bounded stopping times $\tau_1 \geq \tau_0$, and any $A \in \mathcal{F}_{\tau_0}$ we have $1_A(X_{\tau_1} - X_{\tau_0}) \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \setminus L^0_{++}(\Omega, \mathcal{F}, \mathbb{P})$.*

Remark 7 We have $\xi \in L^0(\Omega, \mathbb{F}, P)\setminus L^0_{++}(\Omega, \mathbb{F}, P)$ if and only if $\xi \in L^0(\Omega, \mathcal{F}, Q)\setminus L^0_{++}(\Omega, \mathcal{F}, Q)$ when Q is equivalent to P.

Proof of Proposition 3 *Necessary condition for no arbitrage.* If we assume that $1_A(X_{\tau_1} - X_{\tau_0}) \in L^0_{++}$ for two bounded stopping times $\tau_0 \leq \tau_1$ and $A \in \mathcal{F}_{\tau_0}$, then $1_A 1_{(\tau_0,\tau_1]} \in S^0(\mathbb{F})$ is an arbitrage strategy for X (i.e., no arbitrage implies $1_A(X_{\tau_1} - X_{\tau_0}) \in L^0 \setminus L^0_{++}$).

Sufficient condition for no arbitrage. Assume that for any two bounded stopping times $\tau_0 \leq \tau_1$ and any $A \in \mathcal{F}_{\tau_0}$ we have $1_A(X_{\tau_1} - X_{\tau_0}) \in L^0 \setminus L^0_{++}$ and that X admits arbitrage. Assume that the arbitrage strategy is given by $V = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \in S^0(\mathbb{F})$ with $P(g_j > 0) > 0$ for some $j \in \{1, 2, \ldots, n-1\}$ such that $(V \cdot X)_T \geq 0$ a.s and $P((V \cdot X)_T > 0) > 0$. Let

$$k = \min \left\{ l \in \{0, \ldots, n-1\} : P(g_l > 0) > 0, \quad P\left(\sum_{j=1}^{l} g_j(X_{\tau_{j+1}} - X_{\tau_j}) \geq 0\right) = 1, \right.$$
$$\left. P\left(\sum_{j=1}^{l} g_j(X_{\tau_{j+1}} - X_{\tau_j}) > 0\right) > 0 \right\}.$$

(Note that k is well-defined because we assumed that the arbitrage strategy is given by V.) If $k = 1$ then $P(g_1 > 0) > 0$ and $g_1(X_{\tau_2} - X_{\tau_1}) \geq 0$ a.s and $g_1(X_{\tau_2} - X_{\tau_1}) > 0$ with positive probability. Let $C = \{g_1 > 0\} \in \mathcal{F}_{\tau_1}$. Since $g_1(X_{\tau_2} - X_{\tau_1}) \geq 0$ a.s. we have $X_{\tau_2} \geq X_{\tau_1}$ on C/N in which N is a null set. Because $g_1(X_{\tau_2} - X_{\tau_1}) > 0$ with positive probability we have that $X_{\tau_2} > X_{\tau_1}$ with positive probability on C. As a result $1_C 1_{(\tau_1,\tau_2]} \in L^0_{++}$ which contradicts our assumption. So we assume $k > 1$. From the definition of k, we either have $\sum_{j=1}^{k-1} g_j(X_{\tau_{j+1}} - X_{\tau_j}) = 0$ a.s. or $\sum_{j=1}^{k-1} g_j(X_{\tau_{j+1}} - X_{\tau_j}) < 0$ with positive probability. If $\sum_{j=1}^{k-1} g_j(X_{\tau_{j+1}} - X_{\tau_j}) = 0$ a.s., then $g_k(X_{\tau_{k+1}} - X_{\tau_k}) \geq 0$ a.s. and $P(g_k(X_{\tau_{k+1}} - X_{\tau_k}) > 0) > 0$. If we let $C = \{g_k > 0\}$, we have $1_C 1_{(\tau_k,\tau_{\tau_{k+1}}]} \in L^0_{++}$ which again contradicts our assumption. Let us assume $\sum_{j=1}^{k-1} g_j(X_{\tau_{j+1}} - X_{\tau_j}) < 0$ with positive probability. Let $C = \{\sum_{j=1}^{k-1} g_j(X_{\tau_{j+1}} - X_{\tau_j}) < 0\}$, then $C \in \mathcal{F}_{\tau_{k-1}}$ and $P(C) > 0$ and since $\sum_{j=1}^{k} g_j(X_{\tau_{j+1}} - X_{\tau_j}) \geq 0$ a.s. we have $g_k(X_{\tau_{k+1}} - X_{\tau_k}) > 0$ on C. Since $g_k \geq 0$ a.s., we have $X_{\tau_{k+1}} > X_{\tau_k}$ on C, which implies that $1_C(X_{\tau_{k+1}} - X_{\tau_k}) \in L^0_{++}$. This contradicts our assumption. We can now conclude that X admits no arbitrage with respect to $S^0(\mathbb{F})$. □

The next result the no arbitrage property with respect to $S^0(\mathbb{F})$ is closed under composition with strictly increasing functions.

Proposition 4 *Let $X = (X_t)_{t\geq 0}$ be a Càdlàg stochastic process adapted to the filtration $\mathbb{F}$ and let f be any strictly increasing continuous function whose domain contains* *the range of X. Then the no arbitrage property of X in $S^0(\mathbb{F})$, is equivalent to the no arbitrage property of $Y_t = f(X_t)$ in $S^0(\mathbb{F})$.*

Proof Assume X_t satisfies the no arbitrage property in $S^0(\mathbb{F})$. We need to show for any $\tau_1 \geq \tau_0$ and any $A \in \mathcal{F}_{\tau_0}$ we have $1_A(f(X_{\tau_1}) - f(X_{\tau_0})) \in L^0 \setminus L^0_{++}$ (Lemma 3). Since X has no arbitrage in $S^0(\mathbb{F})$, we have $1_A(X_{\tau_1} - X_{\tau_0}) \in L^0 \setminus L^0_{++}$. This implies that either $X_{\tau_1} = X_{\tau_0}$ a.s. on A or $X_{\tau_1} < X_{\tau_0}$ with positive probability on A. Since f is strictly increasing, we have either $f(X_{\tau_1}) = f(X_{\tau_0})$ a.s. on A or $f(X_{\tau_1}) < f(X_{\tau_0})$ on A with positive probability and this implies $1_A(f(X_{\tau_1}) - f(X_{\tau_0})) \in L^0 \setminus L^0_{++}$. As a result again by applying Proposition 3 we observe that $f(X)$ satisfies the no arbitrage property with respect to $S^0(\mathbb{F})$.

Now assume that $f(X_t)$ satisfies the no arbitrage property. Since $X_t = g(f(X_t))$ for the inverse function g of f, which is strictly increasing, by the same argument as above we know the no arbitrage property of $f(X_t)$ also implies the no arbitrage property of X_t. □

The following example is an application of the above results. This example provides price processes that are not semimartingales and admit an arbitrage within the class $S(\mathbb{F})$ but do not admit arbitrage with respect to $S^0(\mathbb{F})$.

Example 3 Let B_t be a Wiener process. Then $-|B_t|$ is a supermartingale. So it satisfies the no arbitrage property with respect to $S^0(\mathbb{F})$ as a result of Proposition 3. Then by applying Proposition 4, we conclude that the processes $X_t = e^{-|B_t|^{\frac{1}{2n+1}}}$, $n \geq 1$ also have no arbitrage in $S^0(\mathbb{F})$ for any $n \geq 0$. Note that X_t is not a semi-martingale [see Theorem 71 of Protter (2005)].

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